PHYS2016, Workshop 2 (Term 2):

Question 1

Solution:

\[ x^2 + y^2 + \alpha^2 z^2 = R^2, \quad s^2 + \alpha^2 z^2 = R^2, \quad x^2 + y^2 = s^2. \]

Equation of the surface in cylindrical coordinates

\[ f(s, z, \phi) = s^2 + \alpha^2 z^2 - R^2 = 0. \]

\[ \hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{s \hat{s} + \alpha^2 z \hat{z}}{\sqrt{s^2 + \alpha^4 z^2}}. \]

Substituting \( s \) from the surface equation

\[ s = \sqrt{R^2 - \alpha^2 z^2} \]

we get

\[ \hat{n} = \frac{\sqrt{R^2 - \alpha^2 z^2} \hat{s} + \alpha^2 z \hat{z}}{\sqrt{R^2 + \alpha^2(\alpha^2 - 1)z^2}}. \]

Question 2

Solution:

\[ x^2 + y^2 + \alpha^2 z^2 = R^2, \quad s^2 + \alpha^2 z^2 = R^2, \quad x^2 + y^2 = s^2. \]

Equation of the surface in cylindrical coordinates

\[ f(s, z, \phi) = s^2 + \alpha^2 z^2 - R^2 = 0. \]

\[ \hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{s \hat{s} + \alpha^2 z \hat{z}}{\sqrt{s^2 + \alpha^4 z^2}}. \]

Substituting \( s \) from the surface equation

\[ s = \sqrt{R^2 - \alpha^2 z^2} \]

we get

\[ \hat{n} = \frac{\sqrt{R^2 - \alpha^2 z^2} \hat{s} + \alpha^2 z \hat{z}}{\sqrt{R^2 + \alpha^2(\alpha^2 - 1)z^2}}. \]

Question 3

Solution:

For a sphere \( \alpha = 1 \) and

\[ \hat{n} = \frac{\sqrt{R^2 - z^2 \hat{s} + z \hat{z}}}{R} = \frac{\sqrt{x^2 + y^2 \hat{s} + z \hat{z}}}{\sqrt{x^2 + y^2 + z^2}}. \]
Now
\[ \hat{s} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}} \]

and we get
\[ \hat{n} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}} = \hat{r}. \]

For \( \alpha \to 0 \) the spheroid becomes a very long thin vertical sausage and we get
\[ \hat{n} = \hat{s}. \]

For \( \alpha \to \infty \) the spheroid becomes an almost flat disk, since its top and bottom points \( z = \pm R/\alpha \to 0 \) (see the second picture in Question 1). So we expect that the normal \( \hat{n} \) will be pointing up or down the axis \( \hat{z} \). Indeed,
\[
\lim_{\alpha \to \infty} \hat{n} = \lim_{\alpha \to \infty} \frac{\sqrt{x^2 + y^2} \hat{s} + \alpha^2 z \hat{z}}{\sqrt{R^2 + \alpha^2(\alpha^2 - 1)z^2}} = \pm \hat{z}.
\]

**Question 4**

**Solution:** To calculate the area we split spheroid into thin horizontal rings of the width (small but finite !) \( dl \)

The area of the ring is
\[ dA = 2\pi R_{\text{ring}}dl = 2\pi s(z)dl, \quad s(z) = \sqrt{R^2 - z^2}. \]

In principle radius of the ring changes slightly from top to bottom but this is the second order correction to the area. Indeed, radius is multiplied by a small number \( dl \) and we can ignore variations \( ds \) in the ring radius. Now from the Pythagorean theorem
\[
dl = \sqrt{ds^2 + dz^2} = dz\sqrt{1 + \left(\frac{ds(z)}{dz}\right)^2} = dz\sqrt{1 + \left(\frac{ds(z)}{dz}\right)^2} = dz\sqrt{1 + \left(\frac{-\alpha^2 z}{\sqrt{R^2 - \alpha^2 z^2}}\right)^2}.
\]
and
\[ dl = \sqrt{1 + \frac{\alpha^4 z^2}{R^2 - \alpha^2 z^2}} \, dz = \sqrt{\frac{R^2 + \alpha^2 (\alpha^2 - 1) z^2}{R^2 - \alpha^2 z^2}} \, dz. \]

Finally,
\[ dA = 2\pi s(z) \, dl = 2\pi \sqrt{R^2 + \alpha^2 (\alpha^2 - 1) z^2} \, dz \]
and for the total area we have
\[ A = 2\pi \int_{-R/\alpha}^{R/\alpha} \sqrt{R^2 + \alpha^2 (\alpha^2 - 1) z^2} \, dz. \]

For a sphere \( \alpha = 1 \) and we get
\[ A = 2\pi R \int_{-R}^{R} \frac{\sqrt{R^2 + \alpha^2 (\alpha^2 - 1) z^2}}{\alpha} \, dz = 4\pi R^2 \]
as expected.

In fact, you can evaluate the integral for any \( \alpha \). Read further if you are curious enough to see the general answer.

Using Mathematica it is easy to get
\[ \int \sqrt{a^2 + x^2} \, dx = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{1}{2} a^2 \arcsinh \frac{x}{a}. \]

If you differentiate both parts with respect to \( a \) you can easily derive the integrals given in the assignment using a simple formula (prove it !)

\[ \arcsinh x = \log(x + \sqrt{1 + x^2}). \]

Making a change of variable in the integral for the area
\[ z = \frac{1}{\alpha \sqrt{\alpha^2 - 1}} x \]
we obtain
\[ A = \frac{2\pi}{\alpha \sqrt{\alpha^2 - 1}} \int_{-R \sqrt{\alpha^2 - 1}}^{R \sqrt{\alpha^2 - 1}} \sqrt{R^2 + x^2} \, dx = \frac{2\pi}{\alpha \sqrt{\alpha^2 - 1}} \left( \frac{1}{2} R \sqrt{R^2 + x^2} + \frac{1}{2} R^2 \arcsinh \frac{x}{R} \right) \bigg|_{-R \sqrt{\alpha^2 - 1}}^{R \sqrt{\alpha^2 - 1}} \]

After substitution we get
\[ A = 2\pi R^2 \left( 1 + \frac{\arcsinh \sqrt{\alpha^2 - 1}}{\alpha \sqrt{\alpha^2 - 1}} \right). \]

Let us notice that for \( 0 < \alpha < 1 \) we can simply use \( \sqrt{\alpha^2 - 1} = i \sqrt{1 - \alpha^2} \) to get
\[ A = 2\pi R^2 \left( 1 + \frac{\arcsin \sqrt{1 - \alpha^2}}{\alpha \sqrt{1 - \alpha^2}} \right). \]

For \( \alpha \to 0 \) we get \( A \to \infty \) (the area of infinite sausage is infinity), for \( \alpha \to \infty \)
\[ A = 2\pi R^2 \]
which is the are of two flat disks, one on top of another. And for \( \alpha = 1 \) we get \( A = 4\pi R^2 \) using the L’Hopital’s rule.
**Question 5**

**Solution:**
Pressure $p$ is the force per unit area created by electric field.
Now let us calculate the electric field on the $z$ axis.

$$E = \frac{1}{4\pi \epsilon_0} \int \frac{\sigma(r') (r - r')}{|r - r'|^3} d^3r'.$$

The electric field $E$ will be in the $\hat{z}$ direction, so

$$r = z\hat{z}, \quad r' = z'\hat{z} + s(z')\hat{s}, \quad s(z') = \sqrt{R^2 - \alpha^2 z'^2};$$

$$r - r' = (z - z')\hat{z} - \sqrt{R^2 - \alpha^2 z'^2}\hat{s},$$

$$|r - r'|^{-3} = \left((z - z')^2 + R^2 - \alpha^2 z'^2\right)^{-3/2}$$

and

$$d^2r' = 2\pi \sqrt{R^2 + \alpha^2(\alpha^2 - 1)z'^2} dz'$$

from the previous question.

Substituting it all to the formula for $E$ we finally get

$$E = \frac{\sigma}{2\epsilon_0} \int_{-R/\alpha}^{R/\alpha} \frac{(z - z')\sqrt{R^2 + \alpha^2(\alpha^2 - 1)z'^2}}{((z - z')^2 + R^2 - \alpha^2 z'^2)^{3/2}} dz'$$

Consider the case of sphere $\alpha = 1$

$$E = \frac{\sigma R}{2\epsilon_0} \int_{-R}^{R} \frac{(z - z')}{((z - z')^2 + R^2 - z'^2)^{3/2}} dz'$$

Mathematica gives $E = \sigma/\epsilon_0$ for $z > R$ and $E = 0$ for $-R < z < R$ as expected.

Local field at a charge patch is contributed 50% from the patch itself and 50% from all other charges. Therefore,

$$p = \frac{1}{2} \sigma E.$$

**Question 6**

**Solution:**
Again we consider a thin slice of our spheroid and integrate in vertical direction

$$d^3r' = \pi(s(z')^2 dz' = \pi(R^2 - \alpha^2 z'^2)dz'.$$

$$V = \pi \int_{-R/\alpha}^{R/\alpha} (R^2 - \alpha^2 z'^2) dz'$$

For a sphere we have

$$V = \pi \int_{-R}^{R} (R^2 - z'^2) dz' = \pi \left[R^2 z' - \frac{1}{3} z'^3\right]_{-R}^{R} = \frac{4}{3}\pi R^3.$$ 

Let us calculate the electrical field at edge of spheroid at $r = x\hat{x}$ and set $x = R$ later.

$$E = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(r') (r - r')}{|r - r'|^3} d^3r'.$$
\[ dr' = s' ds' d\phi' dz'. \]

\[ r' = z' \hat{z} + s' \hat{s} = s' \cos \phi' \hat{x} + s' \sin \phi' \hat{y} + z' \hat{z}, \]

\[ r - r' = (x - s' \cos \phi') \hat{x} - s' \sin \phi' \hat{y} - z' \hat{z}, \]

\[ |r - r'|^{-3} = ((x - s' \cos \phi')^2 + s'^2 \sin^2 \phi' + z'^2)^{-3/2}. \]

Therefore, the field in \( \hat{x} \) direction is

\[
E = \frac{\rho}{4\pi \epsilon_0} \int_0^{2\pi} d\phi' \int_{-R/\alpha}^{R/\alpha} dz' \int_0^{\sqrt{R^2 - \alpha^2 z'^2}} s' ds' \frac{(x - s' \cos \phi') \hat{x}}{((x - s' \cos \phi')^2 + s'^2 \sin^2 \phi' + z'^2)^{3/2}}.
\]

Setting \( \alpha = 1 \) we get for a sphere

\[
E = \frac{\rho}{4\pi \epsilon_0} \int_0^{2\pi} d\phi' \int_{-R}^{R} dz' \int_0^{\sqrt{R^2 - z'^2}} s' ds' \frac{(x - s' \cos \phi') \hat{x}}{((x - s' \cos \phi')^2 + s'^2 \sin^2 \phi' + z'^2)^{3/2}}.
\]

After substitution \( x = R \) Mathematica gives

\[ E = \frac{\rho R}{3 \epsilon_0} \hat{x}. \]