Incremental Analysis

Small signal trick
Review

The 6.002x world

Linear
- Superposition
  - Thevenin, Norton

Nonlinear
- Analytical
- Graphical

Lumped Circuit Abstraction
- KVL/KCL
- Node
- Composition

Small Signal Model

Digital
- For fixed input values...

For fixed input values...

\[ A = 1 \]
\[ B = 1 \]

\[ V_{IN} \rightarrow A \rightarrow B \rightarrow V_{OUT} \]
Nonlinear Analysis

- Analytical method
- Graphical method
- Piecewise Linear method

Today
- Incremental analysis or small signal method

Reading: Section 4.5
Method 3: Incremental Analysis

(Actually a particular disciplined use of a circuit)

Motivation: music over a light beam. Can we pull this off?

\[ v_I(t) \uparrow \downarrow v_D \]

\[ \text{LED} \quad i_D \quad \text{light intensity} \quad X_D \propto i_D \]

\[ i_R \quad \text{AMP} \quad \text{light intensity} \quad X_R \propto i_R \quad \text{in photoreceiver} \]

\[ v_I \]

\[ \text{music signal} \]

\[ t \]

Remember
LED: Light Emitting expoDweep 😊
Problem: The LED is nonlinear → distortion
Problem: The LED is Nonlinear
Insight

small region looks linear (about some given $V_D, I_D$)
Imagine
How to implement this trick

“Boost and Shrink”
- Shrink signal of interest $v_i$
- Add DC offset to it $V_I$

$V_D = V_D + \Delta D$

$V_I = \Delta i_D + \Delta D$

$\Delta i_D = I_D + \Delta D$

$V_I = V_I + \Delta E$

$V_D = V_D + \Delta D$

$V_I = V_I + \Delta E$

$V_D = I_D + \Delta D$

$V_I = V_I + \Delta E$

$V_D = V_D + \Delta D$

$V_I = V_I + \Delta E$
So, this is really a very clever and disciplined way of using a circuit so that we can get a more or less linear response out of a nonlinear circuit.
The incremental method: (or small signal method)

1. Operate at some DC offset or bias point $V_D, I_D$.
2. Superimpose small signal (music) on top of $V_D$.
3. Response $i_d$ to small signal $v_d$ is approximately linear.

\[ i_d = I_D + i_d \]
What does this mean mathematically?

Or, why is the small signal response linear?

Using Taylor's Expansion to expand \( f(v_D) \) near \( v_D = V_D \):

\[
i_D = f(V_D) + \frac{d f(V_D)}{d V_D} \Delta V_D + \frac{1}{2!} \frac{d^2 f(V_D)}{d V_D^2} \Delta V_D^2 + \cdots
\]

Neglect higher order terms if \( \Delta V_D \) small.
Why is the small signal response linear?

\[ i_D \approx f(V_D) + \frac{df(v_D)}{dv_D} \Delta v_D \]

- constant w.r.t. \( \Delta v_D \)
- slope at \( V_D, I_D \)

\[ i_D = f(V_D) \] is the operating point.
Why is the small signal response linear?

\[ I_D + \Delta i_D \approx f(V_D) + \frac{df(V_D)}{dv_D} \bigg|_{v_D=V_D} \cdot \Delta v_D \]

- **DC:**
  - \( I_D = f(V_D) \)
  - Operating point

**Linear varying part**

\[ \Delta i_D = \frac{df(V_D)}{dv_D} \cdot \Delta v_D \]

\( \frac{df(V_D)}{dv_D} \) is constant w.r.t. \( \Delta v_D \)

\[ \text{so} \quad \Delta i_D \propto \Delta v_D \]

Incremental response is linear
In our example

\[ i_D = f(v_D) = ae^{bv_D} \]

\[ I_D + \Delta i_D \approx f(V_D) + \left( \frac{df(v_D)}{dv_D} \right)_{v_D=V_D} \cdot \Delta V_D \]

From \( \bigcirc \):

\[ I_D + \Delta i_D \approx a e^{bv_D} + a b e^{bv_D} \cdot V_d \]

Equating DC and incremental terms

\[ \begin{align*}
  &D < bV_D \\
  &I_D = a e
\end{align*} \]

Operating point aka DC offset aka bias point

Incremental terms

\[ \Delta I_D = a e - b V_D \]

\[ \Delta I_D = \frac{I_D - b V_D}{v_D} \]

Small signal response is linear!
Graphical interpretation

\[ I_D = a e^{b V_D} \]
\[ i_d = I_D \cdot b \cdot V_D \]

- operating point
- slope at \( V_D, I_D \)

we are approximating \( A \) with \( B \)
We studied the small signal graphically and mathematically. Next, circuit view.

\[ v_I(t) \]

\[ + \]

\[ - \]

\[ v_D \]

\[ + \]

\[ - \]

\[ I_D = a e^{b v_D} \]

\[ i_d = I_D \cdot b \cdot v_d \]

\[ v_D = f(v_D) = a e^{b v_D} \]
A circuit view of the small signal model

Large signal circuit:

\[ V_I \circ \& \ & I_D \circ \& \ & V_D \]

Small signal response:

\[ i_d = I_D \cdot b \cdot v_d \]

For small signals, device behaves like a resistor.

Where did you see \( i = \text{constant} \times v \) before?

\[ R = \frac{1}{I_D \cdot b} \]

\[ \frac{df(v_D)}{dv_D} = v_D = V_D \]
So, We Can Build a Small Signal Circuit

Large signal circuit:

Small signal response:

For small signals, device behaves like a resistor!

See pages 222 - 228 of textbook
Small Signal Circuit Method

1. Find operating point using DC bias inputs from large signal circuit.

2. Develop small signal (linearized) models for each of the elements around the operating point.

Replace original elements with small signal element models.

3. Analyze resulting linearized circuit to obtain small signal response...

Typically involves a nonlinear analysis.

Key: we can use superposition and other linear circuit tools with linearized circuit!
Step 2: Voltage Sources and DC Supply $V_0$

**Large Signal**

$V_S = V_0$

**Small Signal**

$V_S = f(i_S)$

DC voltage source behaves as short to small signals.

DC current source behaves as open to small signals.

See page 416 of textbook.
Current Sources

**Large Signal**

\[ I_S = I_0 \]

**Small Signal**

\[ i_S = f(V_S) = I_0 \]

DC voltage source behaves as short to small signals.

DC current source behaves as open to small signals.
Voltage Source Containing Both DC and Small Signal

large signal

Small signal
Similarly, \( R \)

\[
V_R = RI_R
\]

**large signal**

\[
\begin{align*}
I_R & \quad + \\
V_R & \quad -
\end{align*}
\]

**small signal**

\[
\begin{align*}
in & \quad + \\
\frac{\partial}{\partial V_R} & \quad -
\end{align*}
\]

\[
v_n = R \frac{\partial i_R}{\partial V_R} \Big|_{i_R = I_R}
\]

\[
v_n = R \cdot in
\]
For Non-Linear Device D

large
signal

\[ I_D = a e^{bV_D} \]

small
signal

We will visit small signal circuits again shortly...
Small signal circuit analysis example

Find \( i_D \) for \( v_i \)

Assume \( R = 1 \Omega \), \( a = \frac{1}{4} \text{A} \), \( b = 1 \text{V}^{-1} \)

Also assume that the bias point is at \( V_I = 1 \text{V} \)
Step 1

Analytical method

\[ V_D - V_I + a e = 0 \]
\[ V_D - 1 + \frac{1}{4} e = 0 \]

For \( e = 1 \Omega \)
\[ a = \frac{1}{4} \frac{A}{\Omega} - 1 \]
\[ b = \frac{1V}{a} + V_I = 1V \]

\[ V_D = 0.5V, I_D = 0.44A \]
Step 2

\[ R \xrightarrow{\text{ss model}} \quad \text{ss model} \quad \xrightarrow{\text{ss}} \quad R \quad \text{ss} \quad \text{eqn} \quad \text{ss} \quad ? \]

\[ R_D = \frac{1}{\frac{df(v_D)}{dv_D}} \quad v_D = V_D \]

\[ R_D = \frac{1}{a \cdot b \cdot e \cdot \frac{b v_D}{v_D}} \quad v_D = v \]
Step 3

Small Signal Circuit

\[ i_{d} = \frac{1}{I_{D B}} \]

\[ V_{d} = V_{i} \]

\[ R + R_{D} \]

\[ L = 1 \frac{1}{I_{D B}} \]

from large signal analysis

\[ I_{D} = 0.44 \text{A} \]